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Instabilities in elastic–plastic fluid-saturated porous media: harmonic wave versus acceleration wave analyses

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Abstract

A linearized analysis of the stability of the flow in elastic–plastic fluid-saturated porous media with incompressible constituents is performed. A relationship is established between the results of this analysis and the results of analyses based on acceleration waves performed by Loret and Harireche [*Journal of the Mechanics and Physics of Solids* **39,** 569–606 (1991)] and by Loret et al. [*International Journal of Solids and Structures* **34,** 1583–1608 (1997)]; a justification is found for the growth of the acceleration waves in the non-associative case when their speeds are real and a clarification is provided relative to the growth or decay of waves in the interior of flutter regions. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

An analysis of the nature of the wave-speeds and of the propagation modes of acceleration waves in elastic–plastic fluid-saturated porous media was performed in Loret and Harireche (1991): the conditions for the existence of stationary acceleration waves were derived and it was shown that a non-associative flow rule may lead to the existence of complex squares of wave-speeds, a phenomenon known as *flutter*. In Loret et al. (1997), the growth or decay of acceleration waves was studied. Under simplifying assumptions concerning the material state on the wave-front, it was found that when the squares of the wave-speeds are *real*, non-associative flow rules may give rise to the *growth* in time of the amplitudes of the acceleration waves and also that this growth can become unbounded at the onset of the flutter phenomenon (i.e., when two plastic wave-speeds coalesce), suggesting the formation of a shock. It was also found that the coefficient that was used to characterize the growth or decay of the acceleration waves when their speeds were real would have, inside the flutter region (i.e., when the wave-speeds become *complex*), the sign that char-

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acterized the *decay* of the acceleration waves. This fact was apparently contradictory with the usual interpre- tation of flutter (Rice, 1976) as the *growth* in time of the amplitude of propagation of *harmonic* waves.

In the present paper a linearized analysis of the stability of the flow of elastic–plastic fluid-saturated porous media with incompressible constituents is performed by studying the growth of a small plane harmonic perturbation. The objectives of the present paper are:

(i) to establish, for elastic–plastic fluid-saturated porous media, a relationship between the results of the analysis based on acceleration waves and those of the analysis based on harmonic waves; this relationship involves not only the velocities of propagation of these two kinds of waves but also their rates of growth in time;
(ii) to understand the reasons for the growth of acceleration waves in the non-associative case when their speeds are real; this results from the presence of viscous terms due to Darcy's law in the porous medium, a striking effect being the destabilizing role that viscous terms may play in non-associative problems;
(iii) to clarify the apparent contradiction between the results of Loret et al. (1997) and the interpretation of Rice (1976) in what concerns growth or decay of waves in the interior of the flutter region; the present analysis clarifies in what circumstances the coefficient used in Loret et al. (1997) to characterize the rate of growth of acceleration waves does indeed have that physical meaning.

Notice that the exclusive consideration of the plastic loading regime in the present linearized stability analysis precludes an assessment of the full consequences of the detected flutter instabilities, since the growing oscillatory nature of the flutter solutions leads to situations of local plastic unloading which are quite apart from the linearized loading regime. On the other hand it makes sense to compare the results of the present harmonic wave analysis with those obtained in the acceleration wave analyses of Loret and Harireche (1991) and Loret et al. (1997) because only the linearized constitutive equations for the plastic loading regime were also used in those works.

Harmonic waves in poroelastic materials have been considered by several authors (e.g., Biot, 1956a, b; Atkin, 1968; Bowen and Reinicke, 1978; Bowen and Lockett, 1983; Beskos, 1989). A study on material instability for drained and undrained behavior of an elastic–plastic porous medium has been performed by Molenkamp (1991a, b). The stability of the flow of fluid-saturated inelastic porous media has been investigated for idealized initial-boundary value problems in both quasi-static (Rice, 1975) and dynamic contexts (Vardoulakis, 1986).

The paper is organized as follows. In Section 2, the constitutive equations for an elastic–plastic mixture when both the fluid and the solid constituents are incompressible (Loret and Harireche, 1991) and the equations of linear momentum balance for a saturated two-phase porous material are briefly recalled. In Section 3, the characteristic equation yielding the harmonic wave-speeds is obtained. In Section 4, quasi-explicit expressions for the plastic modulus indicating the onset of divergence and the onset of flutter are obtained and a relationship between the results of the analyses based on acceleration waves and on harmonic waves is established. Some numerical results presented in Simões (1997) suggest that the major qualitative features of the results when both the fluid and the solid constituents are compressible are similar to those of the incompressible case studied in the present paper. Finally, in Section 5, the main conclusions of this study are summarized and discussed.

2. Governing equations

2.1. *Constitutive equations for the elastic–plastic mixture with incompressible constituents*

The rate constitutive equations developed in the framework of the theories of mixtures by Loret and Harireche (1991) relate the rates of the partial stress tensors $\dot{\mathbf{t}}_\alpha$ to the velocity gradients $\partial \mathbf{v}_\alpha/\partial \mathbf{x}$; here and throughout the paper, the Greek indices α and β apply to the solid ($\alpha, \beta = s$) and fluid ($\alpha, \beta = w$) phases or constituents. Each phase of the mixture has a mass M_α and a volume V_α. Let $M = \Sigma_\alpha M_\alpha$ and $V = \Sigma_\alpha V_\alpha$ be, respectively, the total mass and total volume of the mixture. Both solid and fluid phases are inviscid. The fluid is a perfect fluid whose partial pressure is denoted by p^w; thus, $\mathbf{t}^w = -p^w \boldsymbol{\delta}$, with $\boldsymbol{\delta}$ denoting the second-order unit tensor in a 3-D space. The continuum is linear isotropic with respect to its elastic properties but its plastic properties may embody any kind of anisotropy.

If both constituents are incompressible and in absence of chemical reactions, the conservation of mass of both constituents constrains the volumetric strain rates (Bowen, 1976):

$$n^s \operatorname{div} \mathbf{v}_s + n^w \operatorname{div} \mathbf{v}_w = 0. \tag{2.1}$$

The volume fractions $n^\alpha = V_\alpha/V \in]0, 1[$, $\alpha = s, w$, obey the constraint

$$n^s + n^w = 1. \tag{2.2}$$

The elastic material response is characterized by two constants λ_s^* and μ_s which can be related to measurable quantities like the Biot and the Skempton parameters (Bowen, 1982; Loret and Harireche, 1991). The restrictions on the ranges of these parameters (Bowen, 1976),

$$\mu_s > 0, \quad \lambda_s^* + \tfrac{2}{3}\mu_s > 0, \tag{2.3}$$

are known to imply the existence of real and strictly positive elastic acceleration wave-speeds (Loret and Harireche, 1991):

$$c_s^e = \sqrt{\frac{\mu_s}{\rho_s}}, \quad c_L^e{}^* = \sqrt{\frac{1}{r}\frac{\lambda_s^* + 2\mu_s}{\rho_s}}; \tag{2.4}$$

c_s^e is the speed of propagation of the elastic shear acceleration wave (of multiplicity 2), $c_L^e{}^*$ is the speed of propagation of the elastic longitudinal acceleration wave, the scalar r is given by

$$r = 1 + \left(\frac{n^s}{n^w}\right)^2 \frac{\rho^w}{\rho^s} > 1, \tag{2.5}$$

and $\rho^\alpha = M_\alpha/V$, $\alpha = s, w$, are the apparent mass densities.

The rate constitutive equations for the elastic–plastic mixture with incompressible constituents are best expressed in terms of the rate of Terzaghi's effective stress $\mathbf{t}'^s$ defined by

$$\mathbf{t}'^s = \mathbf{t}^s - \frac{n^s}{n^w}\mathbf{t}^w. \tag{2.6}$$

The rate constitutive equations read

$$\dot{\mathbf{t}}'^{s} = \mathscr{A}^{*ss}\colon \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}, \tag{2.7}$$

where the fourth order tensor $\mathscr{A}^{*ss}$, which is endowed with minor symmetries in its two first and two last indices, is a rank-one modification with respect to the elastic constitutive contribution:

$$\mathscr{A}^{*ss} = \mathbf{E}^{*s} - \frac{l^*}{H}(\mathbf{E}^{*s}\colon \mathbf{P}) \otimes (\mathbf{Q}\colon \mathbf{E}^{*s}). \tag{2.8}$$

The isotropic elastic tensor $\mathbf{E}^{*s}$ is given by

$$\mathbf{E}^{*s} = \lambda_s^* \boldsymbol{\delta} \otimes \boldsymbol{\delta} + 2\mu_s \mathbf{I} \tag{2.9}$$

where $\mathbf{I}$ is the fourth-order symmetric unit tensor with components $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. In the above formulas, $\otimes$ denotes a dyadic product, a dot "$\cdot$" (or a double dot ":") denotes a contraction product, so that, in Cartesian components, adjacent indices are summed (or pairwise summed). The corotational terms are neglected in the stress rates so that a superimposed dot denotes the material time derivative which, in the following linearized analysis, is approximated by the partial derivative with respect to time $\partial/\partial t$. $\mathbf{P}$ and $\mathbf{Q}$ are the unit outward normals to the plastic potential and yield surface, respectively, both assumed to be smooth. The loading/unloading index l^* is equal to one for plastic loading, that is, when the stress point is on the yield surface and the plastic index Λ^* is strictly positive,

$$\Lambda^* = \frac{1}{H}\left(\mathbf{Q}\colon \mathbf{E}^{*s}\colon \frac{\partial \mathbf{v}_s}{\partial \mathbf{x}}\right) > 0, \tag{2.10}$$

and it is equal to 0 otherwise. In this linearized analysis we shall assume that *plastic loading* [$l^* = 1$ in eqn (2.8)] *holds pointwise*. The modulus H is assumed to be strictly positive in order to exclude locking behavior:

$$H = h + h_e > 0, \quad h_e = \mathbf{Q}\colon \mathbf{E}^{*s}\colon \mathbf{P}. \tag{2.11}$$

The analysis is restricted to solid skeletons whose plastic dilatant behavior obeys a flow rule that preserves deviatoric associativity, namely

$$\mathbf{P} = \cos\chi \hat{\mathbf{S}} + \frac{\sin\chi}{\sqrt{3}}\boldsymbol{\delta}, \quad \mathbf{Q} = \cos\psi \hat{\mathbf{S}} + \frac{\sin\psi}{\sqrt{3}}\boldsymbol{\delta}, \tag{2.12}$$

where χ is the dilatancy angle, ψ is the friction angle (restricted by the conditions $0 \leqslant \chi \leqslant \psi < \pi/2$) and $\hat{\mathbf{S}}$ is a unit deviatoric tensor in which any kind of induced anisotropy can be embodied. When $\hat{\mathbf{S}}$ is proportional to the deviatoric part of Terzaghi's effective stress $\mathbf{t}'^{s}$ (2.6) the underlying yield surface is of the Drucker–Prager type. The assumption of deviatoric associativity is well supported by experiments performed on many geomaterials (e.g. Baker and Desai, 1982). Notice that, if the solid skeleton obeys an associative flow rule, that is $\mathbf{P} = \mathbf{Q}$, the constitutive equations of the porous medium with incompressible constituents display the major symmetry property.

2.2. *Balance of linear momentum and momentum supplies*

For each phase $\alpha = s, w$ of the porous medium, the balance of momentum

$$\operatorname{div} \mathbf{t}^{\alpha} + \hat{\mathbf{p}}_{\alpha} + \rho^{\alpha}(\mathbf{b}_{\alpha} - \mathbf{a}_{\alpha}) = 0 \quad (\text{no sum over } \alpha) \tag{2.13}$$

involves, in addition to the usual terms present in single phase solids (divergence of the stress tensor, body force per unit mass $\mathbf{b}_{\alpha}$ and acceleration $\mathbf{a}_{\alpha}$), the apparent mass density ρ^{α} and the momentum supply $\hat{\mathbf{p}}_{\alpha}$ to the phase α by the rest of the mixture. Momentum supplies are subject to the constraint

$$\hat{\mathbf{p}}_{s} + \hat{\mathbf{p}}_{w} = 0. \tag{2.14}$$

As in Loret et al. (1997), we assume in this linearized analysis that the momentum supplies include only the so-called Stokes drag effects and, in addition, we adopt simply an isotropic Darcy's law that introduces a single constant material parameter $\xi > 0$, proportional to the inverse of the permeability (Bowen, 1976):

$$\hat{\mathbf{p}}_{s} = -\hat{\mathbf{p}}_{w} = -\xi(\mathbf{v}_{s} - \mathbf{v}_{w}). \tag{2.15}$$

3. The characteristic equation for the harmonic wave-speeds in presence of incompressible constituents

Assuming the body forces $\mathbf{b}_{\alpha}$, $\alpha = s, w$, to be constant in time, we seek solutions to the constitutive eqns (2.7), (2.8) and to the rate form of the equations of linear momentum balance for each phase

$$\operatorname{div} \dot{\mathbf{t}}^{s} + \dot{\hat{\mathbf{p}}}_{s} = \rho^{s} \ddot{\mathbf{v}}_{s},$$

$$\operatorname{div} \dot{\mathbf{t}}^{w} + \dot{\hat{\mathbf{p}}}_{w} = \rho^{w} \ddot{\mathbf{v}}_{w}, \tag{3.1}$$

in the form of plane harmonic waves of assigned wavelength:

$$\mathbf{v}_{\alpha}(\mathbf{x}, t) = \hat{\mathbf{v}}_{\alpha} \exp[ik(\mathbf{n} \cdot \mathbf{x} - ct)], \quad \alpha = s, w, \tag{3.2}$$

where $\hat{\mathbf{v}}_{\alpha}$, $\alpha = s, w$, are undetermined amplitude constants, $\mathbf{n}$ is the unit direction of propagation, k is a real number which represents the angular frequency of the space oscillations ($k = 2\pi/L$, where L is the wavelength) and c is the (possibly complex) speed of propagation. We also assume that the indeterminate rate of the intrinsic pressure in the fluid $\dot{p}^{w}(\dot{\mathbf{t}}^{w} = -\dot{p}^{w}\boldsymbol{\delta})$ is a periodic quantity:

$$\dot{p}^{w} = \dot{p}^{w\#} \exp[ik(\mathbf{n} \cdot \mathbf{x} - ct)]. \tag{3.3}$$

Subtracting eqn $(3.1)_2$ multiplied by n^{s}/n^{w} from eqn $(3.1)_1$ we get

$$\operatorname{div} \dot{\mathbf{t}}'^{s} + \dot{\hat{\mathbf{p}}}_{s} - \frac{n^{s}}{n^{w}} \dot{\hat{\mathbf{p}}}_{w} = \rho^{s} \ddot{\mathbf{v}}_{s} - \frac{n^{s}}{n^{w}} \rho^{w} \ddot{\mathbf{v}}_{w} \tag{3.4}$$

which involves the time rate of Terzaghi's effective stress (2.6). Upon substitution of eqns (3.2), (3.3) and of the constitutive eqns (2.7)–(2.8) in $(3.1)_2$, (3.4) and in the incompressibility condition (2.1), the squares of the wave-speeds are found to be solutions of the problem

$$\begin{bmatrix} \mathbf{A}^{*ss}-\left(ic\dfrac{\xi}{kn^w\rho^s}+c^2\right)\boldsymbol{\delta} & ic\dfrac{\xi}{k(\rho^s\rho^w)^{1/2}}\boldsymbol{\delta} & \dfrac{i}{k(\rho^s)^{1/2}}\dfrac{n^s}{n^w}\mathbf{n} \\ ic\dfrac{\xi}{k(\rho^s\rho^w)^{1/2}}\boldsymbol{\delta} & -\left(ic\dfrac{\xi}{k\rho^w}+c^2\right)\boldsymbol{\delta} & \dfrac{i}{k(\rho^w)^{1/2}}\mathbf{n} \\ \dfrac{i}{k(\rho^s)^{1/2}}\dfrac{n^s}{n^w}\mathbf{n}^T & \dfrac{i}{k(\rho^w)^{1/2}}\mathbf{n}^T & 0 \end{bmatrix}\cdot\begin{Bmatrix}\mathbf{v}_s^{\#}\\ \mathbf{v}_w^{\#}\\ \dot{p}^{w\#}\end{Bmatrix}=\mathbf{0}, \tag{3.5}$$

where

$$\mathbf{A}^{*ss}=\frac{\lambda_s^*+2\mu_s}{\rho_s}\mathbf{n}\otimes\mathbf{n}+\frac{\mu_s}{\rho^s}(\boldsymbol{\delta}-\mathbf{n}\otimes\mathbf{n})-\frac{1}{H}\frac{\mathbf{a}^*\otimes\mathbf{b}^*}{\rho^s}, \tag{3.6}$$

$$\mathbf{a}^*=(\mathbf{E}^{*s}:\mathbf{P})\cdot\mathbf{n}=\lambda_s^*(\operatorname{tr}\mathbf{P})\mathbf{n}+2\mu_s\mathbf{P}\cdot\mathbf{n},$$

$$\mathbf{b}^*=\mathbf{n}\cdot(\mathbf{Q}:\mathbf{E}^{*s})=\lambda_s^*(\operatorname{tr}\mathbf{Q})\mathbf{n}+2\mu_s\mathbf{n}\cdot\mathbf{Q} \tag{3.7}$$

and

$$\mathbf{v}_\alpha^{\#}=(\rho^\alpha)^{1/2}\hat{\mathbf{v}}_\alpha \quad \text{for}\,\alpha=s,w(\text{no sum over}\,\alpha). \tag{3.8}$$

In order to expand the resulting characteristic equation, it is convenient to define a Cartesian coordinate system $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ in such a way that $\mathbf{e}_1=\mathbf{n}$ and $\mathbf{e}_3\cdot\mathbf{b}=0$. Upon expansion, the characteristic equation is cast in the following polynomial form with respect to the normalized wave-speed W:

$$W^2[(iW)^3-\alpha(iW)^2+(iW)-\alpha f]^2F^*(iW)=0,$$

$$F^*(iW)=u_0^*(iW)^5+u_1^*(iW)^4+u_2^*(iW)^3+u_3^*(iW)^2+u_4^*(iW)+u_5^*, \tag{3.9}$$

where

$$W=\frac{c}{c_s^{e\prime}}, \tag{3.10}$$

α is a non-dimensional wavelength

$$\alpha=\frac{1}{kL_C}=\frac{1}{2\pi}\frac{L}{L_C}, \tag{3.11}$$

L_C is a characteristic length-scale introduced by Darcy's law

$$\frac{1}{L_C}=\frac{\xi}{c_s^e}\left(\frac{1}{\rho^s}+\frac{1}{\rho^w}\right), \tag{3.12}$$

and

$$f = \frac{\rho^s}{\rho^s + \rho^w}, \tag{3.13}$$

and the real coefficients u_i^*, $i = 0, 5$, are given in the Appendix. In deriving these coefficients it has been found instrumental to introduce the quantities x and y which depend on the propagation direction

$$x = x(\mathbf{n}) = \frac{(\mathbf{a}^* \cdot \mathbf{n})(\mathbf{b}^* \cdot \mathbf{n})}{2\mu_s \rho^s}, \quad y = y(\mathbf{n}) = \frac{\mathbf{a}^* \cdot \mathbf{b}^* - (\mathbf{a}^* \cdot \mathbf{n})(\mathbf{b}^* \cdot \mathbf{n})}{2\mu_s \rho^s}, \tag{3.14}$$

and the scalar γ defined as

$$\gamma = \left(\frac{c_L^{e*}}{c_s^e}\right)^2. \tag{3.15}$$

For deviatoric associativity as defined by (2.12) and using the definitions (3.7), the coefficients $x(\mathbf{n})$ and $y(\mathbf{n})$ read:

$$\begin{aligned} x(\mathbf{n}) &= -2(c_s^e)^2 \cos\chi \cos\psi (N_\chi + \mathbf{n} \cdot \hat{\mathbf{S}} \cdot \mathbf{n})(N_\psi + \mathbf{n} \cdot \hat{\mathbf{S}} \cdot \mathbf{n}), \\ y(\mathbf{n}) &= 2(c_s^e)^2 \cos\chi \cos\psi \mathbf{n} \cdot \hat{\mathbf{S}} \cdot [\boldsymbol{\delta} - \mathbf{n} \otimes \mathbf{n}] \cdot \mathbf{n} \cdot \hat{\mathbf{S}}, \end{aligned} \tag{3.16}$$

where, due to the constitutive inequalities following (2.12),

$$N_\psi = \sqrt{3} \tan\psi \frac{\lambda_s^* + 2\mu_s/3}{2\mu_s} \geqslant N_\chi = \sqrt{3} \tan\chi \frac{\lambda_s^* + 2\mu_s/3}{2\mu_s} \geqslant 0. \tag{3.17}$$

The coefficient $y(\mathbf{n})$ is always positive (or zero when $\mathbf{n}$ is an eigenvector of $\hat{\mathbf{S}}$) and the coefficient $x(\mathbf{n})$ is negative in the case of an associative flow rule ($\psi = \chi$). It is also possible to perform a spectral decomposition of the unit deviatoric tensor $\hat{\mathbf{S}}$:

$$\hat{\mathbf{S}} = \sum_{i=1}^{3} \hat{S}_i \mathbf{E}_i \otimes \mathbf{E}_i, \tag{3.18}$$

where the eigenvalues $\hat{S}_i$ are expressed in terms of a generalized Lode angle $\mathscr{L}$, $0 \leqslant \mathscr{L} \leqslant \pi/3$,

$$\hat{S}_i = \sqrt{\tfrac{2}{3}} \cos[\mathscr{L} - \tfrac{2}{3}(i-1)\,\pi], \quad i = 1, 2, 3, \tag{3.19}$$

and the eigenvectors $\mathbf{E}_i$ are uniquely defined in the case of distinct eigenvalues $\hat{S}_i$.

Notice that eqn (3.9) has always the root $W = 0$ which corresponds to a trivial solution ($\mathbf{v}_s = \mathbf{v}_w = \mathbf{0}, \dot{p}_w = 0$). Also, one can show that in the limit as $\alpha \to 0$ (the wavelength L or the inverse of the permeability ξ tend to zero) we recover the characteristic equation for the speeds of propagation of the acceleration waves for incompressible constituents [cf eqns (6.10)–(6.12) in Loret and Harireche, 1991].

The characteristic equation for the elastic porous medium with incompressible constituents is recovered if H is set equal to $+\infty$ in (3.9): then the polynomial equation for W is still given by (3.9) but now with

$$F^*(iW) = (iW)^2 - \frac{\alpha(1-f)}{r(n^w)^2}(iW) + \gamma, \tag{3.20}$$

which has complex roots W with strictly negative imaginary parts.

4. Dynamic instabilities in presence of incompressible constituents

For the solutions (3.2) and (3.3) of assigned wavelength that satisfy the field equations we shall delineate five situations depending on the nature of the wave-speed c or equivalently of the normalized wave-speed W (see Fig. 1):

- if W is real we have a solution periodic in time (a *plane wave*);
- if W is a complex number with negative imaginary part the plane wave *decays* in time;
- if W is zero we have a *stationary wave*;
- if W is a purely imaginary positive number the solution explodes as time elapses without oscillating, a *divergence-type growth*;
- if W is a complex number with positive imaginary part the solution grows while oscillating, a *flutter-type growth*.

In our analysis, it is assumed that during a deformation process the modulus H (or equivalently the plastic modulus h) decreases continuously starting from the value $H = +\infty$ (when the material has elastic behavior and the dynamic instabilities are ruled out). As the parameter H^{-1} is increased, we shall look for situations in which divergence or flutter-type growth occurs along at least one direction.

4.1. *Stationary waves and divergence*

Stationary waves are defined as plane waves that do not propagate through the material. Thus, $W = 0$ must be a root of the characteristic eqn (3.9). Setting W equal to zero in (3.9) we obtain the condition for the onset of a stationary wave,

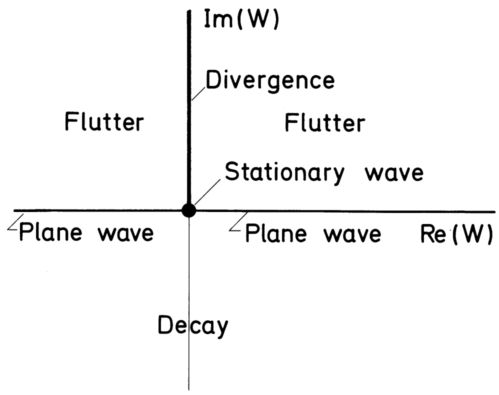


Fig. 1. Sketch of the five possibilities for the nature of the normalized wave-speed W.

$$\left(\frac{H^{sta}}{2\mu_s}\right) = \max_{(\mathbf{n}\in\mathscr{N})} \frac{ry - x/\gamma}{r(c_s^e)^2}, \tag{4.1}$$

where the maximization is to be performed over the set $\mathscr{N}$ such that H is strictly positive. Condition (4.1) coincides with the condition for the onset of a stationary discontinuity (Loret and Harireche, 1991). As expected, the viscous effects due to Darcy's law play no role on the determination of the onset of stationary waves.

Since the solutions W of (3.9) are in general complex, transitions to divergence ($Im(W) > 0, Re(W) = 0$) can occur either when $Re(W) = 0$ and $Im(W) = 0$ (a stationary wave) or when $Im(W) > 0$ and $Re(W) \to 0$. However in the last case divergence occurs necessarily after the occurrence of flutter ($Im(W) > 0$, $Re(W) \neq 0$). We therefore do not investigate the occurrence of these situations in more detail.

4.2. *Flutter*

First of all notice that for elastic mixtures ($H = +\infty$) or for elastic–plastic mixtures with associative plasticity ($\mathbf{P} = \mathbf{Q}$) the matrix in (3.5) is symmetric. Therefore the speeds W are real or complex with negative imaginary parts so that flutter is excluded.

For non-associative plasticity ($\mathbf{P} \neq \mathbf{Q}$) one of the roots of (3.9) may become complex with positive imaginary part leading to an harmonic solution with flutter-type growth. Notice that the first and second factors of (3.9) are also present in the characteristic equation for the elastic case; thus only the plastic wave-speeds that are roots of $F^* = 0$ may become complex leading to flutter.

Looking for transitions to flutter, at which one root of $F^*(iW) = 0$ corresponds to a real wave-speed W, we obtain

$$\left(\frac{H^{flu}}{2\mu_s}\right) = \max_{(\mathbf{n}\in\mathscr{M})} r(c_s^e)^2 \frac{-W^2(ry-x) + \left(\alpha^2 f\dfrac{1-f}{r(n^w)^2}ry + \gamma ry - x\right)}{W^4 - W^2\left(\alpha^2\dfrac{1-f}{r(n^w)^2} + 1 + \gamma\right) + \left(\alpha^2 f\dfrac{1-f}{r(n^w)^2} + \gamma\right)} \tag{4.2}$$

where W^2 satisfies

$$\left(ry - \frac{1}{r(n^w)^2}x\right)W^6 + \left\{\left[\alpha^2\left(\frac{1-f}{r(n^w)^2}\right)^2 - 2\gamma\right]ry - \frac{\alpha^2-2}{r(n^w)^2}x\right\}W^4 + \left(\gamma^2 ry - \frac{1-2f\alpha^2}{r(n^w)^2}x\right)W^2 - \frac{\alpha^2 f^2}{r(n^w)^2}x = 0 \tag{4.3}$$

and the maximization is to be performed over the set $\mathscr{M}$ such that H is strictly positive. In the following sections we study eqns (4.2) and (4.3) in the long wavelength limit and in the short wavelength limit.

4.2.1. *Long wavelength approximation*

For long wavelengths ($\alpha \gg 1$) the transition boundary (4.2) simplifies to

$$\left(\frac{H^{flu}}{2\mu_s}\right) \approx \frac{ry + \varepsilon \dfrac{\sqrt{r(n^w)^2}\sqrt{xry}}{1-f}}{r(c_s^e)^2}, \tag{4.4}$$

where $\varepsilon = \pm 1$. Along this boundary we have

$$W^2 \approx f \frac{\sqrt{x}}{\sqrt{x} + \varepsilon(1-f)\dfrac{\sqrt{ry}}{\sqrt{r(n^w)^2}}}. \tag{4.5}$$

Notice that this transition boundary exists only when $x \geqslant 0$ which may happen only if the flow rule is non-associative. When this is the case, we can prove that

$$H^{flu}_{\varepsilon=1} \geqslant \begin{cases} 0 \geqslant H^{flu}_{\varepsilon=-1}, & \text{for}(1-f)\sqrt{ry} < \sqrt{r(n^w)^2}\sqrt{x}; \\ H^{flu}_{\varepsilon=-1} \geqslant 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Consequently, there exists always at least one transition boundary given by (4.4) with $\varepsilon = +1$ and it precedes the onset of stationary waves given by (4.1).

4.2.2. *Short wavelength approximation*

For short wavelengths ($\alpha <<$) the transition boundary (4.2) simplifies to

$$\left(\frac{H^{flu}}{2\mu_s}\right) \approx -\frac{1}{\tau}\left(\sqrt{x} + \varepsilon\frac{\sqrt{ry}}{\sqrt{r(n^w)^2}}\right)(\sqrt{x} + \varepsilon\sqrt{r(n^w)^2}\sqrt{ry}), \tag{4.7}$$

where $\varepsilon = \pm 1$ and

$$\tau = r[(c_L^e{*})^2 - (c_s^e)^2]. \tag{4.8}$$

Along this boundary we have

$$W^2 \approx \frac{\sqrt{x} + \varepsilon\gamma\sqrt{r(n^w)^2}\sqrt{ry}}{\sqrt{x} + \varepsilon\sqrt{r(n^w)^2}\sqrt{ry}}. \tag{4.9}$$

Again, this transition boundary exists only when $x \geqslant 0$ which may happen only if the flow rule is non-associative.

When $\tau < 0$ we can prove that

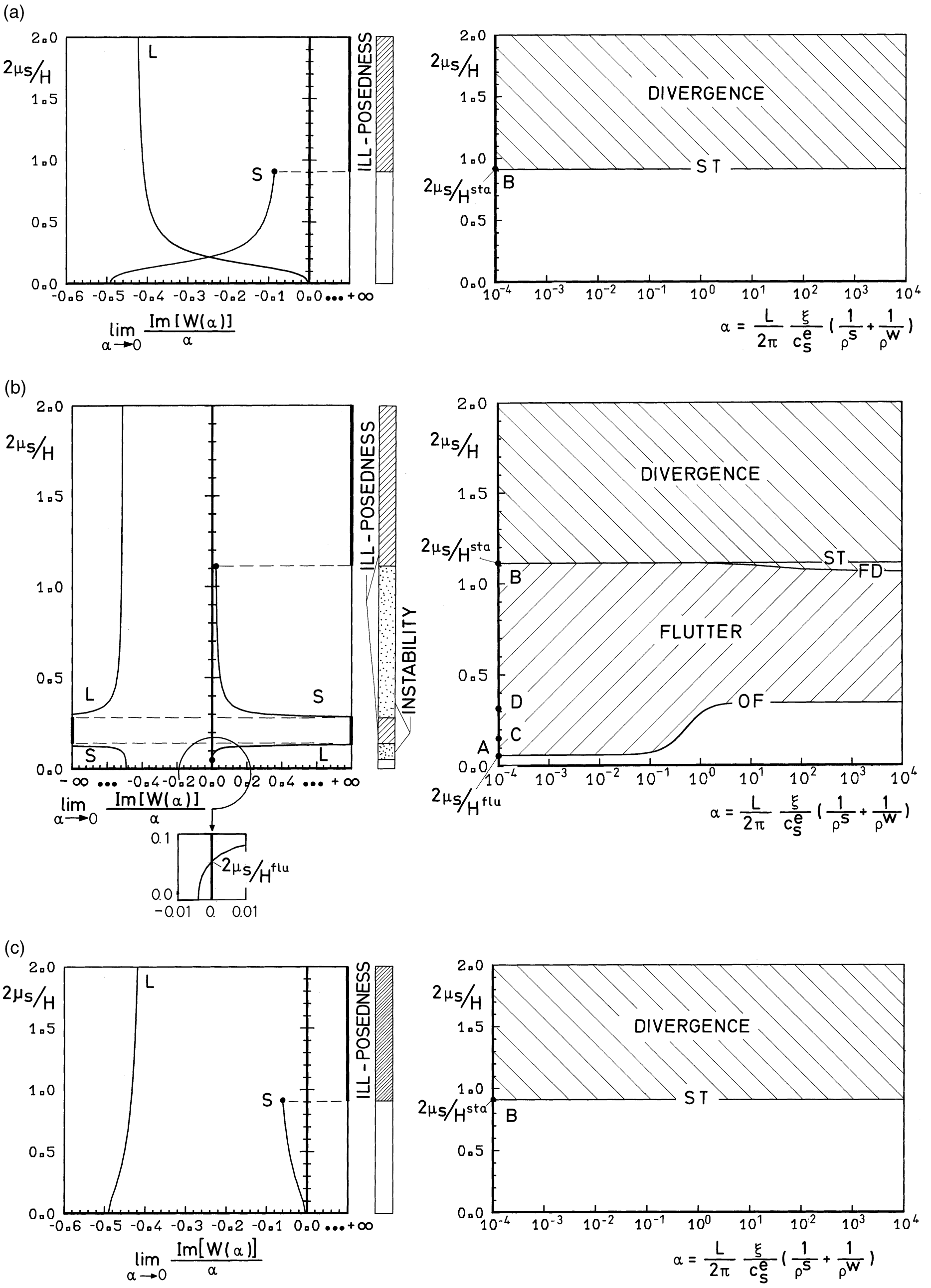


Fig. 2. Porous "material 1" with incompressible constituents and non-associative plasticity. Regions of divergence and flutter growth in the parameter plane $(2\mu_s/H, \alpha)$ and limit as $\alpha \to 0$ of the coefficient $Im(W(\alpha))/\alpha$ which controls the growing time behavior of the harmonic waves. S: Smallest plastic wave-speed. L: Largest plastic wave-speed. ST: Stationary wave $(Re(W) = Im(W) = 0)$. OF: Onset of flutter $(Re(W) \neq 0, Im(W) = 0)$. FD: Flutter-divergence transition $(Re(W) = 0, Im(W) > 0)$. $\chi = 0°$; $\psi = 30°$; $\mathscr{L} = 20°$. (a) $\lambda_s^*/\mu_s = 15$ $(\Rightarrow \tau < 0)$, $\theta = 40°$; (b) $\lambda_s^*/\mu_s = 15$, $\theta = 50°$; (c) $\lambda_s^*/\mu_s = 25$ $(\Rightarrow \tau > 0)$, $\theta = 40°$; (d) $\lambda_s^*/\mu_s = 25$, $\theta = 50°$.

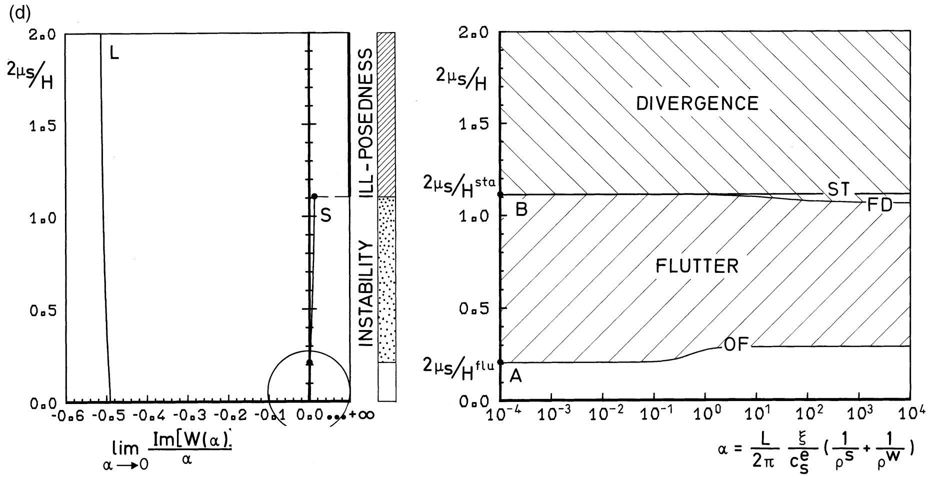


Fig. 2. (*Continued*).

$$H^{flu}_{\varepsilon=1} \geqslant \begin{cases} 0 \geqslant H^{flu}_{\varepsilon=-1}, & \text{for} \sqrt{r(n^w)^2} < \dfrac{\sqrt{ry}}{\sqrt{x}} < \dfrac{1}{\sqrt{r(n^w)^2}}; \\ H^{flu}_{\varepsilon=-1} \geqslant 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

Consequently, there exists always at least one transition boundary given by (4.7) with $\varepsilon = +1$ and it coincides with the boundary that defines the transition from decay to growth in time of the amplitude of the largest acceleration wave in the non-associative elastic–plastic fluid-saturated mixture with incompressible constituents [cf eqn (71) and Fig. 1 in Loret et al., 1997]. When the transition boundary given by (4.7) with $\varepsilon = -1$ exists it coincides with the boundary that defines the transition from growth to decay in time of the amplitude of the smallest acceleration wave [cf eqn (71) and Fig. 2 in Loret et al., 1997]. When $\tau > 0$ we can prove that

$$\left.\begin{array}{ll} \text{for} \sqrt{r(n^w)^2} < \dfrac{\sqrt{ry}}{\sqrt{x}} < \dfrac{1}{\sqrt{r(n^w)^2}}, & H^{flu}_{\varepsilon=-1} \geqslant 0 \\ \text{otherwise,} & 0 \geqslant H^{flu}_{\varepsilon=-1} \end{array}\right\} \geqslant H^{flu}_{\varepsilon=1}. \tag{4.11}$$

Consequently, it may exist one transition boundary given by (4.7) with $\varepsilon = -1$ and, if it exists, it coincides with the boundary that defines the transition from decay to growth in time of the amplitude of the smallest acceleration wave in the non-associative elastic–plastic fluid-saturated mixture with incompressible constituents [cf eqn (71) in Loret et al., 1997].

4.3. *The short wavelength behavior of the harmonic modes*

As seen above, the characteristic equation of the harmonic waves (3.9) reduces to the characteristic equation of the acceleration waves [cf eqns (6.10)–(6.12) in Loret and Harireche, 1991] as $\alpha \to 0$ and then, for given values of μ_s, λ_s^*, ρ^s, ρ^w, n^w, x, y and $H/(2\,\mu_s) > 0$, the solutions $W(\alpha)$ of (3.9)–(3.11) approach the solutions W_A of the characteristic equation of the acceleration waves.

When the solutions W_A are real, the limit as $\alpha \to 0$ of $Im(W(\alpha))$ is equal to $Im(W_A) = 0$ and the limit as $\alpha \to 0$ of the coefficient that controls the growing time behavior of the harmonic waves, $k\, Im(c) = (c_s^e/L_C)\ Im(W)/\alpha$, can be calculated by using l'Hôpital's rule and the continuity of $\mathrm{d}(Im\, W(\alpha))/\mathrm{d}\alpha$ at $\alpha = 0$:

$$\lim_{\alpha\to 0} \frac{Im\, W(\alpha)}{\alpha} = \left[\frac{\mathrm{d}\, Im\, W(\alpha)}{\mathrm{d}\alpha}\right]_{(\alpha=0)}. \tag{4.12}$$

Let us first consider the elastic wave-speeds which are the roots of the second factor of (3.9). By differentiation we get

$$\left(\frac{\mathrm{d}W}{\mathrm{d}\alpha}\right)_{(\alpha=0)} = i\frac{W_A^2 - f}{-3W_A^2+1} \tag{4.13}$$

where W_A is the normalized speed of the elastic shear acceleration wave $W_A = 1$, and then we obtain the result

$$\left[\frac{\mathrm{d}\, Im(W)}{\mathrm{d}\alpha}\right]_{(\alpha=0)} = -\frac{1-f}{2} < 0. \tag{4.14}$$

Let us now consider the plastic wave-speeds. By differentiation of F^* we get

$$\left[\frac{\mathrm{d}\, Im(W)}{\mathrm{d}\alpha}\right]_{(\alpha=0)} = -\frac{1-f}{2}$$
$$\times \frac{\left[r\left(1+\frac{\gamma}{r(n^w)^2}\right) - \frac{r y - x/(r(n^w)^2)}{(H/2\mu_s)(c_s^e)^2}\right]W_A^2 - \left[\gamma r - \frac{\gamma r y - x}{(H/2\mu_s)(c_s^e)^2}\right]\left(1+\frac{1}{r(n^w)^2}\right)}{\left[r(1+\gamma) - \frac{r y - x}{(H/2\mu_s)(c_s^e)^2}\right]W_A^2 - 2\left[\gamma r - \frac{\gamma r y - x}{(H/2\mu_s)(c_s^e)^2}\right]} \tag{4.15}$$

where W_A is the real speed of the acceleration wave. In both cases (4.14) and (4.15) the limit as $\alpha \to 0$ of the coefficient that controls the growth or decay in time of the harmonic waves is found to be equal to the coefficient that controls the growth or decay of the acceleration waves obtained in Loret et al. (1997):

$$\lim_{\alpha\to 0} k\, Im(c) = \frac{c_s^e}{L_C}\lim_{\alpha\to 0}\frac{Im(W)}{\alpha} = \frac{c_s^e}{L_C}\left[\frac{\mathrm{d}\, Im(W)}{\mathrm{d}\alpha}\right]_{(\alpha=0)} = -\frac{c_s^e}{L_C}\frac{1-f}{2}X = -\frac{\xi}{2\rho^s}X \tag{4.16}$$

where X is defined by (cf eqn (44) in Loret et al., 1997)

$$X = \frac{\mathbf{e}^L \cdot \mathbf{E} \cdot \mathbf{e}^R}{\mathbf{e}^L \cdot \mathbf{M} \cdot \mathbf{e}^R} \tag{4.17}$$

where $\mathbf{e}^L$ and $\mathbf{e}^R$ are the left and right eigenvectors for the acceleration wave eigenproblem [cf eqn (30) in Loret et al., 1997] and the matrices $\mathbf{E}$ and $\mathbf{M}$ [defined by eqn (54) and (31) in Loret et al., 1997] characterize viscous and inertia effects, respectively. The equality in eqn (4.16) means that, in the regions of the parameter space where the amplitude of the acceleration waves with real speeds of propagation are found to grow or decay, the corresponding harmonic waves are also found to grow or decay at the same rate when their wavelengths are decreased to zero. The transitions from decay to growth in time observed in Loret et al. (1997) for the acceleration waves coincide with the limit as $\alpha \to 0$ of the flutter boundary for the harmonic waves given by (4.7)–(4.9).

When the solutions W_A of the characteristic equation of the acceleration waves are not real, the limit as $\alpha \to 0$ of $Im(W(\alpha))$ is equal to $Im(W_A) > 0$, so that

$$\lim_{\alpha \to 0} k\, Im(c) = \frac{c_s^e}{L_C} \lim_{\alpha \to 0} \frac{Im(W)}{\alpha} = +\infty. \tag{4.18}$$

The exponentially growing time behavior of the harmonic solutions is unboundedly magni- fied for vanishing small values of the arbitrary wavelengths: the problem becomes (linearly) ill-posed in the sense of the definitions proposed by Schaeffer (1990), Benallal (1992) and Benallal et al. (1993). In those cases, the coefficient X, eqn (4.17) (which was used to characterize the growth or decay of the acceleration waves when their speeds were real) does not have such unbounded behavior. However we can show that, in those cases, we still have

$$\frac{c_s^e}{L_C} \left[\frac{d\, Im(W)}{\mathrm{d}\alpha} \right]_{(\alpha=0)} = -\frac{c_s^e}{L_C} \frac{1-f}{2} Re(X) = -\frac{\xi}{2\rho^s} Re(X). \tag{4.19}$$

In the case of elastic porous media with incompressible constituents eqn (4.19) simplifies to

$$\left[\frac{\mathrm{d}\, Im(W)}{\mathrm{d}\alpha} \right]_{(\alpha=0)} = \begin{cases} -\dfrac{1-f}{2} < 0 & \text{for } W = 1, \\[2ex] -\dfrac{1}{2} \dfrac{1-f}{r(n^w)^2} < 0 & \text{for } W = \gamma, \end{cases} \tag{4.20}$$

which means that, as expected, the amplitudes of the harmonic waves strictly decrease in time as $\alpha \to 0$.

4.4. *Examples*

In order to illustrate the results of the present section, the material termed "material 1" in Loret and Harireche (1991) and Loret et al. (1997) is considered. Its volume fraction is $n^w = 0.02$, its densities are related by $\rho^s/\rho^w = 2.5 n^s/n^w$ and its elastic and plastic properties vary within admissible bounds:

$$0 \leqslant \lambda_s^*/\mu_s < 30, \quad 0 \leqslant \chi \leqslant \psi \leqslant 30^\circ.$$

In Fig. 2 we represent the regions of divergence and flutter growth in the parameter plane $(2\mu_s/H, \alpha)$ for various values of λ_s^*/μ_s and for different directions of propagation **n**. The directions of propagation belong to the plane defined by the stress-eigenvectors associated to the major and minor eigenstresses; the angle made by the normal **n** and the direction of the stress-eigenvectors associated with the major eigenstress is denoted by θ. It can be seen that it is not possible to find, for given normalized plastic modulus and material characteristic length, an infimum wavelength for the unstable solutions.

It can also be seen that the limit as $\alpha \to 0$ of the divergence boundary in the harmonic wave analysis coincides with the onset of stationary waves in the acceleration wave analysis (point B) and that the region of flutter is *discontinuous* as $\alpha \to 0$: the limit as $\alpha \to 0$ of the flutter region in the harmonic wave analysis (line AB) is larger than the flutter region in the acceleration wave analysis (line CD) [note that in case (d) it even happens that flutter is excluded in the acceleration wave analysis]. However, when $\alpha \to 0$, the limit values of the harmonic wave-speeds are in agreement with the results of Loret and Harireche (1991) for the corresponding acceleration wave-speeds: for $2\mu_s/H > 2\mu_s/H^{sta}$ the real part of the smallest plastic wave-speed always vanishes as $\alpha \to 0$; for $2\mu_s/H > 2\mu_s/H^{flu}$ the imaginary part of the complex solution W may vanish or not as $\alpha \to 0$, depending on whether flutter is excluded or not for the corresponding acceleration waves.

In Fig. 2 we also represent the limit as $\alpha \to 0$ of the coefficient $Im(W(\alpha))/\alpha$ that controls the growing time behavior of the harmonic waves. That limit coincides with the coefficient that controls the growth or decay of the acceleration waves with real speed of propagation. It can be seen that, although the characteristic length introduced by Darcy's law does not prevent small wavelength unstable modes, the exponentially growing time behavior of the unstable modes is unboundedly magnified for vanishing small wavelengths only when values of $2\mu_s/H$ are used such that non-real speeds of propagation exist for the corresponding acceleration waves. Consequently, *the existence of non-real speeds of propagation for the acceleration waves corresponds to a* (*linearly*) *ill-posed problem*.

In Fig. 3 we represent, for different values of λ_s^*/μ_s and Lode angles, the limit values as $\alpha \to 0$ of the normalized plastic modulus $H^{flu}/2\mu_s$ at the onset of flutter for the harmonic waves together with the values of the normalized plastic modulus at the onset of flutter for the acceleration waves. It can be seen that the viscous effects may anticipate the occurrence of flutter and, in addition, they make flutter possible even when $c_s^e < c_L^e*$, a condition of pivotal influence to exclude flutter for the acceleration waves.

5. Conclusions

The present analysis and results have features both similar to and distinct from what is obtained with an analysis in terms of acceleration waves.

Indeed, as might be expected, the speeds of propagation of the acceleration waves obtained in Loret and Harireche (1991) are equal to the limit of the wave-speeds provided by the harmonic analysis for infinitely small wavelengths or for infinitely small viscous damping associated to Darcy's law. In addition, the coefficient that controls the growing time behavior of the harmonic waves is equal, in the limit of an infinitely small wavelength, to the decay coefficient of the acceleration waves studied in Loret et al. (1997) *in the cases where the latter have real speeds of propagation*. Obviously, in both analyses, the viscous damping effects due to Darcy's law play no role on the determination of the onset of stationary waves.

However, in the harmonic wave analysis for the non-associative case, the viscous damping effects due to Darcy's law may anticipate the occurrence of flutter relatively to the acceleration wave analysis (the no-damping case) and, in addition, they make flutter possible in situations where it was excluded in the acceleration wave analysis (the no-damping case). Similarly to what may happen in finite dimensional circulatory systems (Ziegler, 1968; Huseyin, 1978), the presence of viscous damping has a destabilizing effect on the behavior of the continuous medium. These facts are not contradictory with those obtained by Loret et al. (1997) for acceleration waves, *provided the speeds of propagation of these are real*, because the presence of those viscous damping effects do play a role in the analysis of the growth or decay of the acceleration waves: in the regions of the parameter space where the amplitudes of the acceleration waves are found to grow or decay, the corresponding harmonic waves are also found to grow or decay at the same rate when their wavelengths are decreased to zero.

Inside the regions where there are some speeds of propagation of acceleration waves that are not real, it is found that the coefficient that controls the time growth or decay of the corresponding harmonic waves becomes unbounded in the limit of infinitely small wavelengths: the problem becomes (linearly) ill-posed in the sense of the definitions proposed by Schaeffer (1990), Benallal (1992) and Benallal et al. (1993). In contrast with this behavior, the coefficient that was used to characterize the growth or decay of the acceleration waves when their speeds were real can still be defined inside the regions of ill-posedness, where it continues to be bounded. However inside those regions of ill-posedness that coefficient cannot have the same physical meaning because those acceleration waves actually do not exist as such: they do not propagate through the medium with a real speed. This resolves the apparent contradiction between the results of Loret et al. (1997) and the interpretation of Rice (1976) in what concerns growth or decay of waves in the interior of the flutter region. The relevance (if any) of the coefficient $Re(X)$ and of the equality (4.19) *inside the region of ill-posedness* is presently unknown to the authors.

In view of the discussion above and restricting ourselves to the linearized problems studied here, in Loret and Harireche (1991) and in Loret et al. (1997), a clear distinction should be made between (linear) ill-posedness and (linear) instability (see also Schaeffer, 1990).

Linear instability is characterized by:

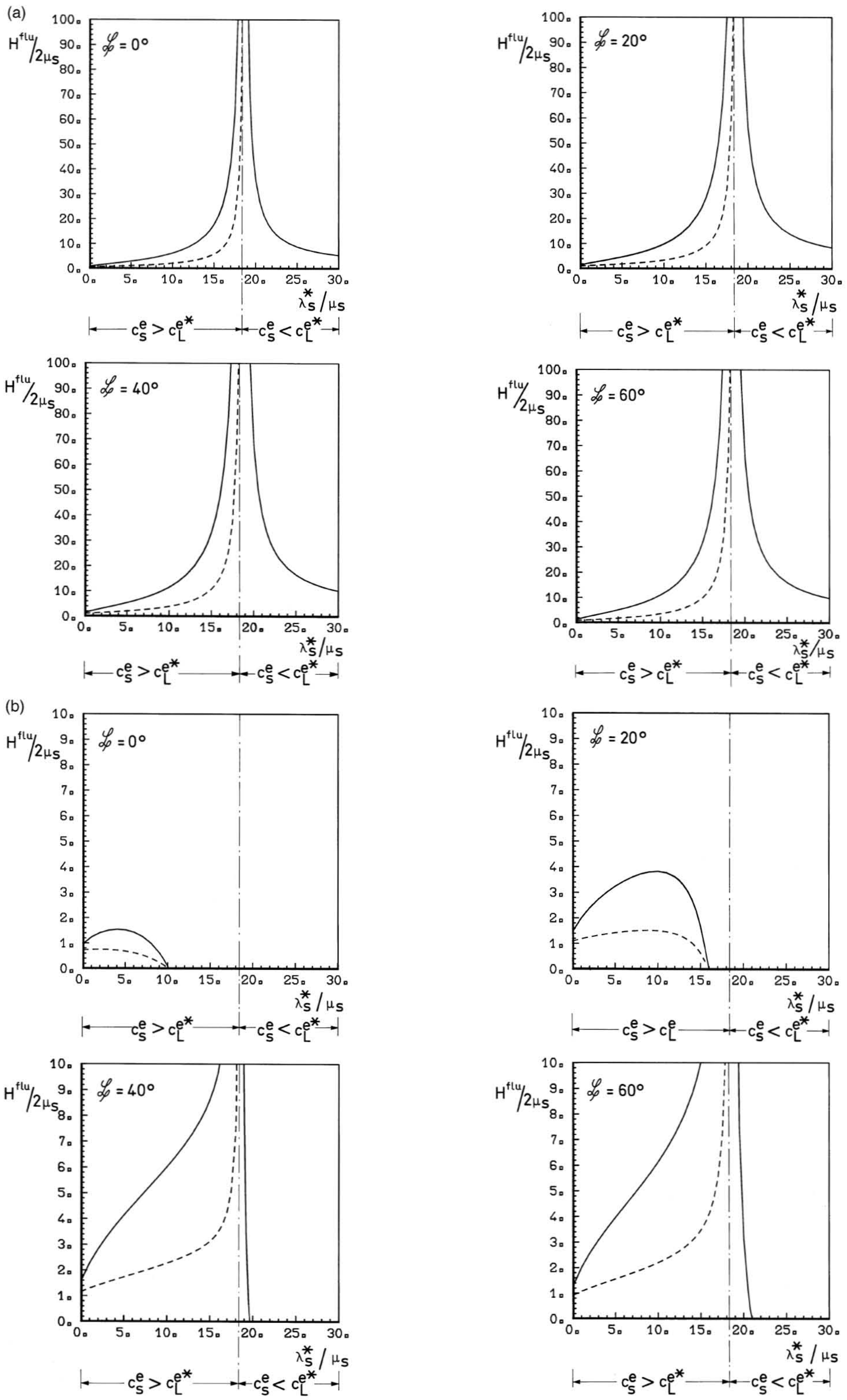
—existence of acceleration waves with a real speed of propagation and with an amplitude that grows exponentially in time;
—existence of harmonic waves growing exponentially in time at a rate which, in the limit as the wavelengths are decreased to zero, equals the rate of growth of the corresponding acceleration waves.

Regions AC and DB in Fig. 2(b) are representative of this situation.

Linear ill-posedness is characterized by:

—existence of non-real wave-speeds for the acceleration wave analysis;
—existence of harmonic wave solutions with an exponential growth in time that is unboundedly magnified as the wavelengths are decreased to zero.

Fig. 3. Porous "material 1" with incompressible constituents and non-associative plasticity. —: Limit values as $\alpha \to 0$ of the normalized plastic modulus $H^{flu}/(2\mu_s)$ at the onset of flutter for different values of λ_s^*/μ_s and of the Lode angle $\mathscr{L}$. – – –: Corresponding values obtained by Loret and Harireche (1991) in the acceleration wave analysis. (a) $\psi = 30°$, $\chi = 0°$; (b) $\psi = 30°$, $\chi = 2.5°$.


(a)
(b)
$H^{flu}/2\mu_s$
$\mathscr{L} = 0°$
$\mathscr{L} = 20°$
$\mathscr{L} = 40°$
$\mathscr{L} = 60°$
λ_s^* / μ_s
$c_s^e > c_L^{e*}$
$c_s^e < c_L^{e*}$

Regions CD and B to ∞ in Fig. 2(b) are representative of this situation.

For an interpretation of flutter ill-posedness as the absence of solution for some kinds of initial conditions see Bigoni and Willis (1994).

The non-coincidence of the regions of instability and ill-posedness in the linearized problems discussed here results from the characteristic length introduced by the viscous term arising from Darcy's law.

Appendix. The coefficients of the characteristic equation (3.9)

The real coefficients of the characteristic eqn (3.9) can be split into elastic and plastic contributions, u_{ie}^* and $u_{ip}^*/(H/2\,\mu_s)$, respectively:

$$u_i^* = u_{ie}^* + u_{ip}^*/(H/2\mu_s), \quad \text{for } i = 0 \text{ to } 5,$$

where

$$u_{0e}^* = -1$$

$$u_{1e}^* = \alpha\left[1 + \frac{1-f}{r(n^w)^2}\right]$$

$$u_{2e}^* = -\left[\alpha^2 \frac{1-f}{r(n^w)^2} + 1 + \gamma\right]$$

$$u_{3e}^* = \alpha\left[f + \frac{1-f}{r(n^w)^2} + \gamma\right]$$

$$u_{4e}^* = -\left[\alpha^2 f \frac{1-f}{r(n^w)^2} + \gamma\right]$$

$$u_{5e}^* = \alpha f \gamma$$

and

$$u_{0p}^* = 0$$

$$u_{1p}^* = 0$$

$$u_{2p}^* = \frac{1}{r(c_s^e)^2}(ry - x)$$

$$u_{3p}^* = -\frac{\alpha}{r(c_s^e)^2}\left[ry\left(f + \frac{1-f}{r(n^w)^2}\right) - x\right]$$

$$u_{4p}^* = \frac{1}{r(c_s^e)^2}\left[\alpha^2 f \frac{1-f}{r(n^w)^2} ry + \gamma ry - x\right]$$

$$u_{5p}^{*} = -\frac{\alpha f}{r(c_s^e)^2}(\gamma r y - x).$$

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